

HAUSDORFF-YOUNG-PALEY INEQUALITIES AND L^p - L^q FOURIER MULTIPLIERS ON LOCALLY COMPACT GROUPS

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ABSTRACT. In this paper we establish the L^p - L^q boundedness of Fourier multipliers on locally compact separable unimodular groups for the range for $1 < p \leq 2 \leq q < \infty$. For this, we also prove a version of the Hausdorff-Young-Paley inequality on general locally compact separable unimodular groups. In particular, the obtained result implies Hörmander's Fourier multiplier theorem on \mathbb{R}^n and the corresponding known results for Fourier multipliers on compact Lie groups.

1. INTRODUCTION

The aim of this paper is to give sufficient conditions for the L^p - L^q boundedness of Fourier multipliers on locally compact separable unimodular groups G .

To put this in context, we recall that in [Hör60, Theorem 1.11], Lars Hörmander has shown that for $1 < p \leq 2 \leq q < \infty$, if the symbol $\sigma_A: \mathbb{R}^n \rightarrow \mathbb{C}$ of a Fourier multiplier A on \mathbb{R}^n satisfies the condition

$$\sup_{s>0} s \left(\int_{\xi \in \mathbb{R}^n: |\sigma(\xi)| \geq s} d\xi \right)^{\frac{1}{p} - \frac{1}{q}} < +\infty, \quad (1.1)$$

then A is a bounded operator from L^p and L^q . Here, as usual, the Fourier multiplier A acts by multiplication on the Fourier transform side, i.e. $\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi)$, $\xi \in \mathbb{R}^n$. Moreover, it then follows that

$$\|A\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int_{\substack{\xi \in \mathbb{R}^n \\ |\sigma_A(\xi)| \geq s}} d\xi \right)^{\frac{1}{p} - \frac{1}{q}}, \quad 1 < p \leq 2 \leq q < +\infty. \quad (1.2)$$

The L^p - L^q boundedness of Fourier multipliers has been also recently investigated in the context of compact Lie groups, and we now briefly recall the result. Let G be a compact Lie group and \widehat{G} its unitary dual. For $\pi \in \widehat{G}$, we write d_π for the dimension of the (unitary irreducible) representation π . In [ANR15, Theorem 3.1] (for a compact Lie group as a special case of a compact homogeneous manifold) the

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authors have shown that, for a Fourier multiplier A acting via $\widehat{A}f(\pi) = \sigma_A(\pi)\widehat{f}(\pi)$ by its global symbol $\sigma_A(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ we have

$$\|A\|_{L^p(G) \rightarrow L^q(G)} \lesssim \sup_{s>0} s \left(\sum_{\substack{\pi \in \widehat{G} \\ \|\sigma_A(\pi)\|_{\text{op}} \geq s}} d_\pi^2 \right)^{\frac{1}{p} - \frac{1}{q}}, \quad 1 < p \leq 2 \leq q \leq \infty. \quad (1.3)$$

For a general development of global symbols and the corresponding global quantization of pseudo-differential operators on compact Lie groups we can refer to [RT13, RT10].

The result of this paper (Theorem 2.15) generalises both multiplier theorems (1.2) and (1.3) to the setting of general locally compact separable unimodular groups G . Namely, in Theorem 2.15 we prove the following inequality

$$\|A\|_{L^p(G) \rightarrow L^q(G)} \lesssim \sup_{s>0} s \left[\int_{t \in \mathbb{R}_+ : \mu_t(A) \geq s} dt \right]^{\frac{1}{p} - \frac{1}{q}}, \quad 1 < p \leq 2 \leq q < +\infty, \quad (1.4)$$

where $\mu_t(A)$ are the t -th generalised singular values of A , see [TK86] (and also below) for definition and properties. The proof of inequality (1.4) is based on a version of the Hausdorff-Young-Paley inequality on locally compact separable group G that we establish for this purpose. The latter inequality is formulated in Theorem 2.12 and proved later in Section 3. In Section 2 we recall some von Neumann algebras constructions that become instrumental in our proof and simplify the exposition.

In Remark 2.16 and Proposition 2.18 we show that the multiplier theorem (1.4) implies both (1.2) and (1.3) in the respective settings of \mathbb{R}^n and compact Lie groups.

The assumption for the locally compact group to be separable and unimodular may be viewed as natural allowing one to use basic results of von Neumann-type Fourier analysis, such as, for example, Plancherel formula (see Segal [Seg50]). However, the unimodularity assumption may be in principle avoided, see e.g. [DM76], but the exposition becomes much more technical. For a more detailed discussion of pseudo-differential operators in such settings we refer to [MR15], but we note that in this paper we do not need to assume that the group is, for example, of type I or type II.

Some results on L^p -Fourier multipliers in the spirit of Hörmander-Mihlin theorem are also known on groups. See, for example, [CW71] for the case of the group $\text{SU}(2)$, [RW15] for general compact Lie groups, and [FR14] for graded Lie groups. The approach to the L^p -Fourier multipliers is different from the technique proposed in this paper allowing us to avoid making an assumption that the group is compact or nilpotent. In this paper we are interested in Fourier multipliers rather than in spectral multipliers (see e.g. [CGM93], and also [Cow74]), and for the discussion of some relations between those in the group setting we can refer to [RW15] and references therein. Finally we note that multiplier estimates on noncommutative groups are in general considerably more delicate than those in the commutative case, recall e.g. the asymmetry problem and its resolution in [DGR00].

2. MAIN RESULTS

In this section we introduce the necessary notation and formulate our results.

2.1. Notation and preliminaries. In this section we give preliminaries on von Neumann algebras to be used for harmonic analysis on locally compact groups. For exposition purposes it seems beneficial to recall several general notions in the context of general von Neumann algebras M . However, for our application to multipliers on locally compact groups G we will be later setting M to be the right group von Neumann algebra $\text{VN}_R(G)$. In particular, we will be able to readily apply the notion of noncommutative Lorentz spaces on M as developed in [Kos81].

Let $M \subset \mathcal{L}(\mathcal{H})$ be a semifinite von Neumann algebra acting in a Hilbert space \mathcal{H} with a trace τ . The semifinite assumption simplifies the formulations and is satisfied in our main example $M = \text{VN}_R(G)$.

Definition 2.1. An operator A (possibly unbounded in \mathcal{H}) is said to be *affiliated with* M , symbolically $A \nu M$, if it commutes with the elements of the commutant M^\dagger of M , i.e.

$$AU = UA, \quad \text{for all } U \in M^\dagger. \quad (2.1)$$

This relation ν is a natural relaxation of \in : if A is a bounded operator affiliated with M , then by the double commutant theorem $A \in M$. One of the original motivations [MvN36, MvN37] of John von Neumann was to build a mathematical foundation for quantum mechanics. In this framework, the observables with unbounded spectrum correspond to closed densely defined unbounded operators. Although the algebra M consists primarily of bounded operators, the technique of projections makes it possible to approximate unbounded operators.

Definition 2.2. A closeable operator A (possibly unbounded) affiliated with M is said to be τ -measurable if for each $\varepsilon > 0$ there exists a projection p in M such that $p\mathcal{H} \subset D(A)$ and $\tau(I - p) \leq \varepsilon$. Here $D(A)$ is the domain of A in \mathcal{H} . We denote by $S(M)$ the set of all τ -measurable operators.

Example 2.3. Let $M = \{M_\varphi: L^2(X, \mu) \ni f \mapsto M_\varphi f = \varphi f \in L^2(X, \mu)\}_{\varphi \in L^\infty(X, \mu)}$ and take $\tau(M_\varphi) := \int_X \varphi d\mu$, where (X, μ) is a measure space. Then an operator M_φ is τ -measurable if and only if φ is a μ -almost everywhere finite function.

The $*$ -algebra $S(M)$ is a basic construction for the noncommutative integration. Let $A = U|A|$ be the polar decomposition. The spectral theorem yields that

$$|A| = \int_{\text{Sp}(|A|)} \lambda dE_\lambda(|A|), \quad (2.2)$$

where $\{E_\lambda(|A|)\}_{\lambda \in \text{Sp}(|A|)}$ is the spectral projections associated with $|A|$. Here $dE_\lambda(|A|)$ should be understood as the relative dimension function first constructed in [MvN36]. Since A is affiliated with M , the projections $E_\lambda(|A|) \in M$. Now, we are ready to ‘measure the speed of decay’ of the operator A .

Definition 2.4. For an operator $A \in S(M)$, define the distribution $d_\lambda(A)$ function by

$$d_\lambda(A) := \tau(E_{(\lambda, +\infty)}(|A|)), \quad \lambda \geq 0, \quad (2.3)$$

where $E_{(\lambda, +\infty)}(|A|)$ is the spectral projection of $|A|$ corresponding to the interval $(\lambda, +\infty)$. For any $t > 0$, we define the generalised t -th singular numbers by

$$\mu_t(A) := \inf\{\lambda \geq 0 : d_\lambda(A) \leq t\}. \quad (2.4)$$

Example 2.5. For the operator M_φ in Example 2.3, from Definition 2.4 we can exhibit its generalised t -th singular numbers to be

$$\mu_t(M_\varphi) = \varphi^*(t),$$

where $\varphi^*(t)$ is the classical function rearrangement (see e.g. [BS88]).

As a noncommutative extension [Kos81] of the classical Lorentz spaces, we define Lorentz spaces $L^{p,q}(M)$ associated with a semifinite von Neumann algebra as follows:

Definition 2.6. For $1 \leq p < \infty$, $1 \leq q \leq \infty$, denote by $L^{p,q}(M)$ the set of all operators $A \in S(M)$ satisfying

$$\|A\|_{L^{p,q}(M)} := \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} \mu_t(A) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty. \quad (2.5)$$

For $q = \infty$, we define

$$\|A\|_{L^{p,\infty}(M)} := \sup_{t>0} t^{\frac{1}{p}} \mu_t(A). \quad (2.6)$$

With this, for $1 \leq p < \infty$, we can also define L^p -spaces by

$$\|A\|_{L^p(M)} := \|A\|_{L^{p,p}(M)} = \left(\int_0^{+\infty} \mu_t(A)^p dt \right)^{\frac{1}{p}}.$$

The classical Lorentz spaces $L^{p,q}(X, \mu)$ correspond to the case of commutative von Neumann algebra. Modulus technical details [Dix81, p. 132, Theorem 1], an arbitrary abelian von Neumann algebra in a Hilbert space \mathcal{H} is isometrically isomorphic to the algebra $\{M_\varphi\}_{\varphi \in L^\infty(X, \mu)}$ for some measure space (X, μ) .

Example 2.7 (Classical Lorentz spaces). Let M be the algebra $\{M_\varphi\}_{\varphi \in L^\infty(X, \mu)}$ from Example 2.3 consisting of all the multiplication operators $M_\varphi : L^2(X, \mu) \ni f \mapsto M_\varphi f = \varphi f \in L^2(X, \mu)$. By Example 2.5, we have

$$\mu_t(M_\varphi) = \varphi^*(t).$$

Thus, the Lorentz space $L^{p,q}(M)$ consists of all operators M_φ such that

$$\int_0^{+\infty} [t^{\frac{1}{p}} \varphi^*(t)]^q \frac{dt}{t} < +\infty,$$

which gives the classical Lorentz space.

The notion of τ -measurability does not appear in the classical theory of Schatten classes since for $M = \mathcal{L}(H)$ we have $S(\mathcal{L}(H)) = \mathcal{L}(H)$.

Concerning the structure of semifinite von Neumann algebras, given an arbitrary semifinite von Neumann algebra M with a trace τ , there is an isomorphism of M onto a certain Hilbert algebra \mathcal{U} ([Dix81, p.99, Theorem 2]). Thus, we construct the trace on the Hilbert algebra yielding the trace on M due to isomorphism. We refer to [Dix81], [Naj72] for more details on this.

Let now G be a locally compact unimodular separable group. Denote by $\pi_L(g)$ and $\pi_R(g)$ the left and the right action of G on $L^2(G)$, respectively:

$$\begin{aligned}\pi_L(g)f(x) &:= f(g^{-1}x), \\ \pi_R(g)f(x) &:= f(xg),\end{aligned}$$

and by $\text{VN}_L(G)$ the group von Neumann algebra generated by all the $\pi_L(g)$ with $g \in G$, i.e.

$$\text{VN}_L(G) := \{\pi_L(g)\}_{g \in G}^{\text{!!}},$$

and similiary

$$\text{VN}_R(G) := \{\pi_R(g)\}_{g \in G}^{\text{!!}},$$

where !! is the bicommutant of the self-adjoint subalgebras $\{\pi_L(g)\}_{g \in G}, \{\pi_R(g)\}_{g \in G} \subset \mathcal{L}(L^2(G))$. It has been shown in [Seg49] that

$$\text{VN}_L(G)^! = \text{VN}_R(G), \quad (2.7)$$

$$\text{VN}_R(G)^! = \text{VN}_L(G). \quad (2.8)$$

We do not make assumption that G is either of type I or type II. The decomposition theory for unitary representations of locally compact separable unimodular groups has been established in [Ern62, Ern61].

From now on we take $M = \text{VN}_R(G)$.

For $f \in L^1(G) \cap L^2(G)$, we say that f on G has a *Fourier transform* whenever the convolution operator

$$R_f h(x) := (h * f)(x) = \int_G h(g)f(g^{-1}x) dg \quad (2.9)$$

is a τ -measurable operator with respect to $\text{VN}_R(G)$, i.e. $R_f \in S(\text{VN}_R(G))$. The Plancherel identity takes ([Seg50, Theorem 3 on page 282]) the form

$$\|R_f\|_{L^2(\text{VN}_R(G))} = \|f\|_{L^2(G)}. \quad (2.10)$$

In this setting, the Hausdorff-Young inequality has been established in [Kun58] in the form

$$\|R_f\|_{L^{p'}(\text{VN}_R(G))} \leq \|f\|_{L^p(G)}, \quad 1 < p \leq 2. \quad (2.11)$$

In [Kos81], as an application of the technique of the t -th generalised singular values, the Hardy-Littlewood theorem ([HL27]) has been generalised to an arbitrary locally compact separable unimodular group G :

Theorem 2.8 ([Kos81]). *Let $1 < p \leq 2$ and $f \in L^p(G)$. Then we have*

$$\|R_f\|_{L^{p',p}(\text{VN}_R(G))} \leq \|f\|_{L^p(G)}. \quad (2.12)$$

Remark 2.9. The Plancherel equality (2.10) by [Seg50] and Kosaki's version [Kos81] of Hardy-Littlewood inequality (2.12) have been established for the left convolution $L_f h = f * h$. However, the same line of reasoning yields inequality (2.12) with the right convolution R_f .

Using the technique of the t -th generalised singular values developed in [TK86], we can formulate both the Hausdorff-Young (2.11) and Hardy-Littlewood (2.12) inequalities in the forms (for $1 < p \leq 2$):

$$\left(\int_0^{+\infty} \mu_t(R_f)^{p'} dt \right)^{\frac{1}{p'}} \equiv \|R_f\|_{L^{p'}(\text{VN}_R(G))} \leq \|f\|_{L^p(G)}, \quad (2.13)$$

$$\left(\int_0^{+\infty} t^{p-2} \mu_t(R_f)^p dt \right)^{\frac{1}{p}} \equiv \|R_f\|_{L^{p',p}(\text{VN}_R(G))} \leq \|f\|_{L^p(G)}. \quad (2.14)$$

2.2. Paley and Hausdorff-Young-Paley inequalities. The results formulated in the sequel will be all proved in Section 3. Our analysis of L^p - L^q multipliers will be based on a version of the Hausdorff-Young-Paley that we establish in the context of locally compact groups. We start first with an inequality that can be regarded as a Paley type inequality.

Theorem 2.10 (Paley inequality). *Let G be a locally compact unimodular separable group. Let $1 < p \leq 2$. Suppose that a positive function $\varphi(t)$ satisfies the condition*

$$M_\varphi := \sup_{s>0} s \int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \geq s}} dt < +\infty. \quad (2.15)$$

Then we have

$$\left(\int_0^{+\infty} \mu_t(R_f)^p \varphi^{2-p}(t) dt \right)^{\frac{1}{p}} \leq M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G)}. \quad (2.16)$$

As usual, the integral over an empty set in (2.15) is assumed to be zero.

We note that taking $\varphi(t) = \frac{1}{t}$ we recover Kosaki's Hardy-Littlewood inequality (2.14). In this sense, the Paley inequality can be viewed as an extension of (one of) the Hardy-Littlewood inequalities. As a small byproduct of our proof of Theorem 2.10 we thus get a simple proof of Theorem 2.8.

Further, we recall a result on the interpolation of weighted spaces from [BL76]:

Theorem 2.11 (Interpolation of weighted spaces). *Let $d\mu_0(x) = \omega_0(x)d\mu(x)$, $d\mu_1(x) = \omega_1(x)d\mu(x)$, and write $L^p(\omega) = L^p(\omega d\mu)$ for the weight ω . Suppose that $0 < p_0, p_1 < \infty$. Then*

$$(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta,p} = L^p(\omega),$$

where $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\omega = \omega_0^{\frac{1-\theta}{p_0}} \omega_1^{\frac{\theta}{p_1}}$.

From this, interpolating between the Paley-type inequality (2.16) in Theorem 2.10 and Hausdorff-Young inequality (2.13), we readily obtain an inequality that will be crucial for our consequent analysis of L^p - L^q multipliers:

Theorem 2.12 (Hausdorff-Young-Paley inequality). *Let G be a locally compact unimodular separable group. Let $1 < p \leq b \leq p' < \infty$. If a positive function $\varphi(t)$, $t \in \mathbb{R}_+$, satisfies condition*

$$M_\varphi := \sup_{s>0} s \int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \geq s}} dt < \infty, \quad (2.17)$$

then we have

$$\left(\int_{\mathbb{R}_+} \left(\mu_t(R_f) \varphi(t)^{\frac{1}{b} - \frac{1}{p'}} \right)^b dt \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(G)}. \quad (2.18)$$

This reduces to the Hausdorff-Young inequality (2.13) when $b = p'$ and to the Paley inequality in (2.16) when $b = p$.

2.3. L^p - L^q boundedness of Fourier multipliers. Let G be a locally compact separable unimodular group. In the first instance we adopt the following definition of Fourier multipliers:

Definition 2.13. A linear operator A is said to be a left *Fourier multiplier* on G if $A \in S(\text{VN}_R(G))$.

If we now recall Definition 2.1 we can see that A is a left Fourier multiplier on G if and only if A is affiliated with the right group von Neumann algebra $\text{VN}_R(G)$ and is τ -measurable. We can then clarify Definition 2.13 further:

Remark 2.14. For $M = \text{VN}_R(G)$ the operators affiliated with M are precisely those A that are left-invariant on G , namely,

$$A \text{ is affiliated with } \text{VN}_R(G) \iff A\pi_L(g) = \pi_L(g)A, \text{ for all } g \in G. \quad (2.19)$$

A simple argument for this will be given in Section 3. Summarising this observation with Definition 2.13, (left) Fourier multipliers on G are precisely the left-invariant operators that are measurable (in the sense of Definition 2.2).

In the following statements, to unite the formulations, we adopt the convention that the sum or the integral over an empty set is zero, and that $0^0 = 0$.

Theorem 2.15. *Let $1 < p \leq 2 \leq q < +\infty$ and suppose that A is a Fourier multiplier on a locally compact separable unimodular group G . Then we have*

$$\|A\|_{L^p(G) \rightarrow L^q(G)} \lesssim \sup_{s>0} s \left[\int_{t \in \mathbb{R}_+ : \mu_t(A) \geq s} dt \right]^{\frac{1}{p} - \frac{1}{q}}. \quad (2.20)$$

For $p = q = 2$ inequality (2.20) is sharp, i.e.

$$\|A\|_{L^2(G) \rightarrow L^2(G)} = \sup_{t \in \mathbb{R}_+} \mu_t(A). \quad (2.21)$$

Using the noncommutative Lorentz spaces $L^{r,\infty}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \neq q$, we can also write (2.20) as

$$\|A\|_{L^p(G) \rightarrow L^q(G)} \lesssim \|A\|_{L^{r,\infty}(VN_R(G))}. \quad (2.22)$$

Remark 2.16. As a special case of $G = \mathbb{R}^n$, Theorem 2.15 implies the Hörmander multiplier estimate (1.2) established in [Hör60, p. 106, Theorem 1.11], and we have

$$\|A\|_{L^{r,\infty}(VN_R(\mathbb{R}^n))} = \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)}. \quad (2.23)$$

This will be shown in Section 3.

In the case of a compact Lie group G , Theorem 2.15 gives a better result than the known estimate (1.3), namely:

Theorem 2.17 ([ANR15, Theorem 3.1]). *Let $1 < p \leq 2 \leq q < \infty$ and suppose that A is a Fourier multiplier on the compact Lie group G . Then we have*

$$\|A\|_{L^p(G) \rightarrow L^q(G)} \lesssim \sup_{s \geq 0} s \left(\sum_{\xi \in \widehat{G}: \|\sigma_A(\xi)\|_{\text{op}} \geq s} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}}. \quad (2.24)$$

The fact that Theorem 2.15 implies Theorem 2.17 follows from the following result relating the noncommutative Lorentz norm to the global symbol of invariant operators in the context of compact Lie groups:

Proposition 2.18. *Let $1 < p \leq 2 \leq q < \infty$ and let $p \neq q$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Suppose G is a compact Lie group and A is a Fourier multiplier on G . Then we have*

$$\|A\|_{L^{r,\infty}(VN_R(G))} \leq \sup_{s > 0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\text{op}} \geq s}} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}}, \quad (2.25)$$

where $\sigma_A(\xi) = \xi^*(g)A\xi(g)|_{g=e} \in \mathbb{C}^{d_\xi \times d_\xi}$ is the global symbol of A .

The global symbolic calculus for operators A acting on compact Lie groups has been introduced and consistently developed in [RT13, RT10], to which we refer to further details on global symbols on compact Lie groups. Here we also note that with this matrix global symbol, the Fourier multiplier acts by $\widehat{A}f(\xi) = \sigma_A(\xi)\widehat{f}(\xi)$, $\xi \in \widehat{G}$.

Remark 2.19. If G is a compact Lie group, the sufficient condition (2.24) on the Fourier multiplier A implies τ -measurability of A with respect to $VN_L(G)$.

Proof of Remark 2.19. The condition of τ -measurability does not arise in the setting of compact Lie groups due to the fact [Ter81, Proposition 21, p. 16] that

$$A \text{ is } \tau \text{ - measurable with respect to } M$$

if and only if

$$\lim_{\lambda \rightarrow +\infty} d_\lambda(A) = 0. \quad (2.26)$$

In the view of Proposition 2.18, the latter condition holds. Indeed, by Definition 2.6 and Proposition 2.18, we get

$$\begin{aligned} \sup_{s>0} s[d_s(A)]^{\frac{1}{r}} &= \sup_{t>0} t^{\frac{1}{r}} \mu_t(A) \\ &= \|A\|_{L^{r,\infty}(VN_R(G))} \leq \sup_{s>0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\| \geq s}} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}} < +\infty, \end{aligned} \quad (2.27)$$

where in the first equality we used (3.5) with $\alpha = \frac{1}{r}$ from Proposition 3.1. Thus, we have

$$d_s(A) \leq \frac{C}{s^r}. \quad (2.28)$$

As a consequence, we obtain (2.26). This completes the proof. \square

3. PROOFS

In this section we prove the results from the previous section.

Proof of Remark 2.14. \implies . By Definition 2.13, we have

$$AU = AU, \quad \text{for all } U \in VN_R(G)^!. \quad (3.1)$$

Since $VN_R(G)^! = VN_L(G)$ by (2.8), and we can take $U = \pi_L(g)$, $g \in G$, we see that A must be left-invariant.

\Leftarrow . We have

$$A\pi_L(g) = \pi_L(g)A, \quad \text{for all } g \in G.$$

By definition, the algebra $VN_L(G)$ is the closure of the involutive subalgebra

$$\{\pi_L(g)\}_{g \in G} \subset \mathcal{L}(L^2(G))$$

in the strong operator topology. Therefore, we obtain

$$AU = UA, \quad \text{for all } U \in VN_R(G)^!. \quad (3.2)$$

This completes the proof. \square

For the sake of the exposition clarity, we now formulate some properties of the distribution function d_A which we will be using in the proofs.

Proposition 3.1. *Let $A \in S(M)$. Then we have*

$$d_A(\mu_A(t)) \leq t; \quad (3.3)$$

$$\mu_A(t) > s \quad \text{if and only if} \quad t < d_A(s); \quad (3.4)$$

$$\sup_{t>0} t^\alpha \mu_A(t) = \sup_{s>0} s[d_A(s)]^\alpha \quad \text{for } 0 < \alpha < \infty. \quad (3.5)$$

The proof of this proposition is almost verbatim to the proof of [Gra08, Proposition 1.4.5 on page 46]. The word ‘almost’ stands for the right-continuity of the non-commutative distribution function $d_A(s)$ which is discussed after [TK86, Definition 1.3 on page 272]. Therefore, in the following proof we shall use the right-continuity of $d_A(s)$ without any justification.

Proof of Proposition 3.1. Let $s_n \in \{s > 0: d_A(s) \leq t\}$ be such that $s_n \searrow \mu_A(t)$. Then $d_A(s_n) \leq t$ and the right-continuity of d_A implies that $d_A(\mu_A(t)) \leq t$. This proves (3.3). Now, we apply this property to derive (3.4). If $s < \mu_A(t) = \inf\{s > 0: d_A(s) \leq t\}$, then s does not belong to the set $\{s > 0: d_A(s) \leq t\} \implies d_A(s) > t$. Conversely, if for some t and s , we had $\mu_A(t) < s$, then the application of d_A and property (3.4) would yield the contradiction $d_A(s) \leq d_A(\mu_A(t)) \leq t$. Property (3.4) is established. Finally, we show (3.5). Given $s > 0$, pick ε satisfying $0 < \varepsilon < s$. Property (3.4) yields $\mu_A(d_A(s) - \varepsilon) > s$ which implies that

$$\sup_{t>0} t^\alpha \mu_A(t) \geq (d_A(s) - \varepsilon)^\alpha \mu_A(d_A(s) - \varepsilon) > (d_A(s) - \varepsilon)^\alpha s. \quad (3.6)$$

We first let $\varepsilon \rightarrow 0$ and then take the supremum over all $s > 0$ to obtain one direction. Conversely, given $t > 0$, pick $0 < \varepsilon < \mu_A(t)$. Property (3.4) yields that $d_A(\mu_A(t) - \varepsilon) > t$. This implies that $\sup_{s>0} s(d_A(s))^\alpha \geq (\mu_A(t) - \varepsilon)(d_A(\mu_A(t) - \varepsilon))^\alpha > (\mu_A(t) - \varepsilon)t^\alpha$. We first let $\varepsilon \rightarrow 0$ and then take the supremum over all $t > 0$ to obtain the opposite direction of (3.5). \square

We recall the following result which will be partially used.

Theorem 3.2 ([Seg53, Theorem 4, p. 412]). *If operators A and B are τ -measurable with respect to a von Neumann algebra M , then so are A^* , $A + B$ and AB , i.e. the maps*

$$+ : M \times M \ni (A, B) \mapsto A + B \in M, \quad (3.7)$$

$$\cdot : M \times M \ni (A, B) \mapsto AB \in M, \quad (3.8)$$

$$* : M \ni A \mapsto A^* \in M \quad (3.9)$$

are well-defined.

Here we formulate some properties of μ_t that we will use in the proof of Theorem 2.15.

Lemma 3.3 ([TK86, Lemma 2.5, p. 275]). *Let A, B be τ -measurable operators. Then the following properties hold true.*

- (1) *The map $(0, +\infty) \ni t \mapsto \mu_t(A)$ is non-increasing and continuous from the right. Moreover,*

$$\lim_{t \rightarrow 0} \mu_t(A) = \|A\| \in [0, +\infty]. \quad (3.10)$$

$$(2) \quad \mu_t(A) = \mu_t(A^*). \quad (3.11)$$

$$(3) \quad \mu_{t+s}(AB) \leq \mu_t(A)\mu_s(B). \quad (3.12)$$

In Lemma 3.3, we formulate only the properties we use, whereas in [TK86, Lemma 2.5, p. 275] the reader can find more details. We now prove Remark 2.16, namely, that our theorem implies Hörmander's theorem on L^p - L^q Fourier multipliers on \mathbb{R}^n .

Proof of Remark 2.16. Indeed, we identify the algebra $\text{VN}_R(\mathbb{R}^n)$ via the Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ with the algebra $Z = \{M_\varphi\}_{\varphi \in L^\infty(\mathbb{R}^n)}$ of the multiplication operators

$$M_\varphi : L^2(\widehat{\mathbb{R}^n}) : h \mapsto M_\varphi h = \varphi h \in L^2(\widehat{\mathbb{R}^n}).$$

Given an element A of $\text{VN}_R(\mathbb{R}^n)$ which acts on $L^2(\mathbb{R}^n)$ by the convolution with its Schwartz kernel K_A ,

$$A: L^2(\mathbb{R}^n) \ni f \mapsto Af = K_A * f,$$

we associate with A the multiplication operator M_{σ_A} acting on $L^2(\widehat{\mathbb{R}}^n)$ via the multiplication by the symbol $\sigma_A = \widehat{K_A}$,

$$M_{\sigma_A}: L^2(\widehat{\mathbb{R}}^n) \ni \widehat{f} \mapsto M_{\sigma_A}\widehat{f} = \sigma_A(\xi)\widehat{f}(\xi) \in L^2(\widehat{\mathbb{R}}^n).$$

Then by Example 2.5 with $\varphi(\xi) = \sigma_A(\xi)$, we get

$$\mu_t(M_{\sigma_A}) = \sigma_A^*(t). \quad (3.13)$$

Thus, Definition 2.6 yields

$$\begin{aligned} \|M_{\sigma_A}\|_{L^{r,\infty}(\text{VN}_R(G))} &= \sup_{t>0} t^{\frac{1}{r}} \mu_t(A) = \sup_{t>0} t^{\frac{1}{r}} \sigma_A^*(t) \\ &= \sup_{t>0} s [d_{M_{\sigma_A}}(s)]^{\frac{1}{p}-\frac{1}{q}} = \sup_{t>0} s \left(\int_{\substack{\xi \in \mathbb{R}^n \\ |\sigma_A(\xi)| \geq s}} d\xi \right)^{\frac{1}{p}-\frac{1}{q}}, \end{aligned} \quad (3.14)$$

where in the equality between the first and the second lines we used Proposition 3.1 with $M = \text{VN}_R(\mathbb{R}^n)$. This completes the proof. \square

We now prove Paley type inequality in Theorem 2.10.

Proof of Theorem 2.10. Let ν give measure $\varphi^2(t)$ to the set consisting of the single point $\{t\}$, $t \in \mathbb{R}_+$, i.e.

$$\nu(t) := \varphi^2(t) dt. \quad (3.15)$$

We define the corresponding space $L^p(\mathbb{R}_+, \nu)$, $1 \leq p < \infty$, as the space of complex (or real) valued functions $f = f(t)$ such that

$$\|f\|_{L^p(\mathbb{R}_+, \nu)} := \left(\int_{\mathbb{R}_+} |f(t)|^p \varphi^2(t) dt \right)^{\frac{1}{p}} < \infty. \quad (3.16)$$

We will show that the sub-linear operator

$$T: L^p(G) \ni f \mapsto Tf := \mu_t(R_f)/\varphi(t) \in L^p(\mathbb{R}_+, \nu)$$

is well-defined and bounded from $L^p(G)$ to $L^p(\mathbb{R}_+, \nu)$ for $1 < p \leq 2$. In other words, we claim that we have the estimate

$$\|Tf\|_{L^p(\mathbb{R}_+, \nu)} = \left(\int_{\mathbb{R}_+} \left(\frac{\mu_t(R_f)}{\varphi(t)} \right)^p \varphi^2(t) dt \right)^{\frac{1}{p}} \lesssim M_{\varphi}^{\frac{2-p}{p}} \|f\|_{L^p(G)}, \quad (3.17)$$

which would give (2.16), and where we set $M_{\varphi} := \sup_{t>0} t \int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \geq s}} dt$. We will show that

T is of weak type (2,2) and of weak-type (1,1). More precisely, with the distribution

function ν , we show that

$$\nu(y; Tf) \leq \left(\frac{M_2 \|f\|_{L^2(G)}}{y} \right)^2 \quad \text{with norm } M_2 = 1, \quad (3.18)$$

$$\nu(y; Tf) \leq \frac{M_1 \|f\|_{L^1(G)}}{y} \quad \text{with norm } M_1 = M_\varphi, \quad (3.19)$$

where ν is defined in (3.15). Recall that the distribution function $\nu(y; Tf)$ with respect to the weight φ^2 is defined as follows

$$\nu(y; Tf) = \int_{\substack{t \in \mathbb{R}_+ \\ \mu_t(R_f) \geq y}} \varphi^2(t) dt.$$

Then (3.17) would follow by the Marcinkiewicz interpolation theorem.

Now, to show (3.18), using Plancherel's identity (2.10), we get

$$\begin{aligned} y^2 \nu(y; Tf) &\leq \|Tf\|_{L^p(\mathbb{R}_+, \nu)}^2 = \int_{\mathbb{R}_+} \left(\frac{\mu_t(R_f)}{\varphi(t)} \right)^2 \varphi^2(t) dt \\ &= \int_{\mathbb{R}_+} \mu_t^2(R_f) dt = \|R_f\|_{L^2(VN_R(G))}^2 = \|f\|_{L^2(G)}^2. \end{aligned}$$

Thus, T is of type (2,2) with norm $M_2 \leq 1$. Further, we show that T is of weak-type (1,1) with norm $M_1 = M_\varphi$; more precisely, we show that

$$\nu\{t \in \mathbb{R}_+ : \frac{\mu_t(R_f)}{\varphi(t)} > y\} \lesssim M_\varphi \frac{\|f\|_{L^1(G)}}{y}. \quad (3.20)$$

The left-hand side here is the integral $\int \varphi^2(t) dt$ taken over those $t \in \mathbb{R}_+$ for which $\frac{\mu_t(R_f)}{\varphi(t)} > y$. From the definition of the Fourier transform it follows that

$$\mu_t(R_f) \leq \|f\|_{L^1(G)}. \quad (3.21)$$

Indeed, from the Definition 2.4, we have

$$\mu_t(R_f) \leq \|R_f\|_{L^2(G) \rightarrow L^2(G)}.$$

The Young inequality for convolution [Fol95, p.52, Proposition 2.39] yields

$$\|R_f g\|_{L^2(G)} \leq \|f\|_{L^1(G)} \|g\|_{L^2(G)}.$$

Thus

$$\|R_f\|_{L^2(G) \rightarrow L^2(G)} \leq \|f\|_{L^1(G)}.$$

This proves (3.21). Therefore, we have

$$y < \frac{\mu_t(R_f)}{\varphi(t)} \leq \frac{\|f\|_{L^1(G)}}{\varphi(t)}.$$

Using this, we get

$$\left\{ t \in \mathbb{R}_+ : \frac{\mu_t(R_f)}{\varphi(t)} > y \right\} \subset \left\{ t \in \mathbb{R}_+ : \frac{\|f\|_{L^1(G)}}{\varphi(t)} > y \right\}$$

for any $y > 0$. Consequently,

$$\nu \left\{ t \in \mathbb{R}_+ : \frac{\mu_t(R_f)}{\varphi(t)} > y \right\} \leq \nu \left\{ t \in \mathbb{R}_+ : \frac{\|f\|_{L^1(G)}}{\varphi(\pi)} > y \right\}.$$

Setting $v := \frac{\|f\|_{L^1(G)}}{y}$, we get

$$\nu \left\{ t \in \mathbb{R}_+ : \frac{\mu_t(R_f)}{\varphi(t)} > y \right\} \leq \int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \leq v}} \varphi^2(t) dt. \quad (3.22)$$

We now claim that

$$\int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \leq v}} \varphi^2(t) dt \lesssim M_\varphi v. \quad (3.23)$$

Indeed, first we notice that we have

$$\int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \leq v}} \varphi^2(t) dt = \int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \leq v}} dt \int_0^{\varphi^2(t)} d\tau.$$

We can interchange the order of integration to get

$$\int_{\substack{t \in \mathbb{R}_+ \\ \varphi(t) \leq v}} dt \int_0^{\varphi^2(t)} d\tau = \int_0^{v^2} d\tau \int_{\substack{t \in \mathbb{R}_+ \\ \tau^{\frac{1}{2}} \leq \varphi(t) \leq v}} dt.$$

Further, we make a substitution $\tau = s^2$, yielding

$$\int_0^{v^2} d\tau \int_{\substack{t \in \mathbb{R}_+ \\ \tau^{\frac{1}{2}} \leq \varphi(t) \leq v}} dt = 2 \int_0^v s ds \int_{\substack{s \in \mathbb{R}_+ \\ s \leq \varphi(t) \leq v}} dt \leq 2 \int_0^v s ds \int_{\substack{t \in \mathbb{R}_+ \\ s \leq \varphi(t)}} dt.$$

Since

$$s \int_{\substack{t \in \mathbb{R}_+ \\ s \leq \varphi(t)}} dt \leq \sup_{s > 0} s \int_{\substack{t \in \mathbb{R}_+ \\ s \leq \varphi(t)}} dt = M_\varphi$$

is finite by the assumption that $M_\varphi < \infty$, we have

$$2 \int_0^v t dt s \int_{\substack{t \in \mathbb{R}_+ \\ s \leq \varphi(t)}} dt \lesssim M_\varphi v.$$

This proves (3.23) and hence also (3.20). Thus, we have proved inequalities (3.18) and (3.19). Then by using the Marcinkiewicz interpolation theorem with $p_1 = 1$,

$p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\left(\int_{\mathbb{R}_+} \left(\frac{\mu_t(R_f)}{\varphi(t)} \right)^p \varphi^2(t) dt \right)^{\frac{1}{p}} = \|Af\|_{L^p(\mathbb{R}_+, \nu)} \lesssim M_{\varphi}^{\frac{2-p}{p}} \|f\|_{L^p(G)}.$$

This completes the proof. \square

We now prove our L^p - L^q Fourier multiplier theorem.

Proof of Theorem 2.15. Since the algebra $S(\text{VN}_R(G))$ of left Fourier multipliers A is closed under taking the adjoint $S(\text{VN}_R(G)) \ni A \mapsto A^* \in S(\text{VN}_R(G))$ (see [Seg53, Theorem 4, p. 412] or [Ter81, Theorem 28 on p. 4]), and

$$\|A\|_{L^p(G) \rightarrow L^q(G)} = \|A^*\|_{L^{q'}(G) \rightarrow L^{p'}(G)} \quad (3.24)$$

we may assume that $p \leq q'$, for otherwise we have $q' \leq (p')' = p$ and use (3.11) that $\mu_t(A^*) = \mu_t(A)$. When $f \in L^p(G)$, dualising the Hausdorff-Young inequality (2.13) gives, since $q' \leq 2$,

$$\|Af\|_{L^q(G)} \leq \left(\int_0^{+\infty} [\mu_t(R_{Af})]^{q'} dt \right)^{\frac{1}{q'}}. \quad (3.25)$$

In order for the group convolution to be associative, we may restrict to the Bruhat-Schwartz space $\mathcal{D}(G)$. This space and its properties have been developed by Bruhat in [Bru61] as an analogue of the the space of compactly supported smooth functions in the setting of locally compact groups. Similar to the usual space of test functions, this space is dense in the L^p spaces for $1 \leq p < \infty$ and in its dual space of distributions in the corresponding topologies, so there is no loss of generality restricting the operators to such spaces since then the required estimates follow by the density and standard approximation argument. We also refer to [MR15] for a brief account of the construction of this space and its main properties.

By the Schwartz kernel theorem in Bruhat-Schwartz spaces [Bru61, Corollaire 2, p. 56] it follows that the operator A has the integral kernel. Consequently, the left-invariance (see Remark 2.14) implies that A is a convolution operator.

The associativity of the convolution [Bru61, Sec. 6, page 56] yields

$$R_{Af} = AR_f, \quad f \in \mathcal{D}(G).$$

By our assumptions, A and R_f are measurable with respect to $\text{VN}_R(G)$. This makes it possible to apply [TK86, Lemma 2.5 on page 275] to obtain the estimate

$$\mu_t(R_{Af}) = \mu_t(AR_f) \leq \mu_t(A)\mu_t(R_f). \quad (3.26)$$

Thus, we obtain

$$\|Af\|_{L^q(G)} \leq \left(\int_0^{+\infty} [\mu_t(A)\mu_t(R_f)]^{q'} dt \right)^{\frac{1}{q'}}. \quad (3.27)$$

Now, we are in a position to apply the Hausdorff-Young-Paley inequality in Theorem 2.12. With $\varphi(t) = \mu_t(A)^r$ for $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, the assumptions of Theorem 2.12 are then satisfied, and since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, we obtain

$$\left(\int_0^{+\infty} [\mu_t(R_f)\mu_t(A)]^{q'} dt \right)^{\frac{1}{q'}} \leq \sup_{s>0} \left[s \int_{\substack{t \in \mathbb{R}_+ \\ \mu_t(A)^r \geq s}} dt \right]^{\frac{1}{r}} \|f\|_{L^p(G)}. \quad (3.28)$$

Further, it can be easily checked that

$$\left(\sup_{s>0} s \int_{\substack{t \in \mathbb{R}_+ \\ \mu_t(A)^r \geq s}} dt \right)^{\frac{1}{r}} = \left(\sup_{s>0} s^r \int_{\substack{t \in \mathbb{R}_+ \\ \mu_t(A) \geq s}} dt \right)^{\frac{1}{r}} = \sup_{s>0} s \left(\int_{\substack{t \in \mathbb{R}_+ \\ \mu_t(A) \geq s}} dt \right)^{\frac{1}{r}}. \quad (3.29)$$

This completes the proof. \square

Proof of Proposition 2.18. We first compute the norm $\|A\|_{L^{r,\infty}(VN_R(G))}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \neq q$. By definition, we have

$$\|A\|_{L^{r,\infty}(VN_R(G))} = \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \mu_A(t). \quad (3.30)$$

The application of the property (3.5) from Proposition 3.1 yields

$$\sup_{t>0} t^{\frac{1}{p}} \mu_A(t) = \sup_{s>0} s [d_A(s)]^{\frac{1}{p}-\frac{1}{q}}.$$

Therefore, it is sufficient to show that

$$\sup_{s>0} s [d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\| \geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \quad (3.31)$$

The polar decomposition for arbitrary closed densely defined possibly unbounded operators A acting on a Hilbert space \mathcal{H} has been established in [vN32]. Thus, we apply [vN32, page 307, Theorem 7] to get

$$A = W|A|, \quad (3.32)$$

where W is a partial isometry. This means that the operators W^*W and WW^* are projections in \mathcal{H} . If A is a left Fourier multiplier, then its modulus $|A|$ is affiliated with $VN_R(G)$ as well:

Lemma 3.4 ([MvN36, p. 33, Lemma 4.4.1]). *Let M be a von Neumann algebra. Suppose A is affiliated with M . Then $|A|$ is affiliated with M as well and $W \in M$.*

To proceed, we will use the following property:

Claim 3.5. *Let $A \in S(\text{VN}_R(G))$ and $E_{[s, +\infty)}(|A|)$ be the spectral measure of $|A|$ corresponding to the interval $[s, +\infty)$. Then we have*

$$d_A(s) = \sum_{\xi \in \widehat{G}} d_\xi \sum_{\substack{n=1, \dots, d_\xi \\ s_{n,\xi} \geq s}} 1, \quad (3.33)$$

where for fixed $n = 1, \dots, d_\xi$, the number $s_{n,\xi}$ is the joint eigenvalue for the eigenfunctions ξ_{kn} , $k = 1, \dots, d_\xi$, of $|A|$. These functions ξ_{kn} , $k = 1, \dots, d_\xi$, generate the subspace $\mathcal{H}^{n,\xi} = \text{span}\{\xi_{kn}\}_{k=1}^{d_\xi}$.

We note that by Remark 2.14, in view of the left invariance of the operators A and $|A|$, by the Peter-Weyl theorem they leave the spaces $\mathcal{H}^{n,\xi}$ invariant (for the discussion of the spaces $\mathcal{H}^{n,\xi}$ in the context of the Peter-Weyl theorem we refer to [RT10, Theorem 7.5.14 and Remark 7.5.16]).

Assuming Claim 3.5 for the moment, the proof proceeds as follows. Without loss of generality, we can reorder, for each $\xi \in \widehat{G}$, the numbers $s_{n,\xi}$ putting them in a decreasing order with respect to $n = 1, \dots, d_\xi$ (thus, also reordering the corresponding eigenfunctions). Then we can estimate

$$d_A(s) = \sum_{\xi \in \widehat{G}} d_\xi \sum_{\substack{n=1, \dots, d_\xi \\ s_{n,\xi} \geq s}} 1 \leq \sum_{\xi \in \widehat{G}} d_\xi \sum_{\substack{n=1, \dots, d_\xi \\ s_{1,\xi} \geq s}} 1 = \sum_{\substack{\xi \in \widehat{G} \\ s_{1,\xi} \geq s}} d_\xi^2,$$

where in the first inequality we used the fact

$$\{\xi \in \widehat{G}, n = 1, \dots, d_\xi : s_{n,\xi} \geq s\} \subset \{\xi \in \widehat{G}, n = 1, \dots, d_\xi : s_{1,\xi} \geq s\} \quad (3.34)$$

since for fixed $\xi \in \widehat{G}$ the sequence $\{s_{n,\xi}\}_{n=1}^{d_\xi}$ monotonically decreases. We notice that

$$s_{1,\xi} = \|\sigma_A(\xi)\|_{\text{op}}.$$

Thus, we obtain

$$d_A(s) \leq \sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\text{op}} \geq s}} d_\xi^2.$$

From this, we get

$$s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\text{op}} \geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \quad (3.35)$$

Taking supremum in the right-hand side of (3.35), we get

$$s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\text{op}} \geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \quad (3.36)$$

Then taking again the supremum in the left-hand side of (3.36), we finally obtain

$$\sup_{s>0} s[d_A(s)]^{\frac{1}{p}-\frac{1}{q}} \leq \sup_{s>0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\text{op}} \geq s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}.$$

This proves (3.31). Now, we justify the claim (3.33).

Since G is compact, its von Neumann algebra $M = VN_R(G)$ is a type I factor. The trace τ for type I factors M (and we denote it by Tr in this case) can be given [Naj72, page 478] by

$$\text{Tr}(A) = \int_{-\infty}^{+\infty} \lambda dD_M(E_\lambda), \quad A \in M, \quad (3.37)$$

where

$$D_M: M_+ \rightarrow \{1, 2, 3, \dots\} \quad (3.38)$$

is the dimension function introduced in [MvN36, MvN37] and M_+ is the set of all hermitian ($A^* = A$) operators $A \in M$. For each value of λ the projection E_λ is the sum of minimal mutually orthogonal projection operators, hence the value $D_M(E_\lambda)$ can increase only in jumps and its points of growth s_n are the characteristic values of the operator A . Thus, we get

$$\text{Tr}(A) = \sum_{n \in \mathbb{N}} m_n s_n, \quad (3.39)$$

where m_n is the corresponding jump of the function $D_M(E_\lambda)$. Further, we determine the singular values of A , or equivalently we will look for the eigenvalues of $|A|$.

Indeed, we recall that $|A| \big|_{\bigoplus_{k=1}^{d_\xi} \mathcal{H}^{k,\xi}} = \sigma_{|A|}(\xi)$ and use the fact that $s_{1,\xi} = \|\sigma_{|A|}(\xi)\|_{\text{op}}$. It is convenient to enumerate the singular values $s_{k,\xi}$ by two elements (k, ξ) , $k = 1, \dots, d_\xi$, in view of the decomposition into the closed subspaces invariant under the group action. We rewrite (3.39) once again as the usual trace

$$\text{Tr}(|A|) = \sum_{\pi \in \widehat{G}} d_\pi \sum_{n=1}^{d_\pi} s_{n,\pi}, \quad (3.40)$$

where we write $s_{n,\pi}$ for the eigenvalue of the restriction $|A| \big|_{\bigoplus_{n=1}^{d_\pi} \mathcal{H}^{n,\pi}}$ of $|A|$ to the subspaces $\mathcal{H}^{k,\pi}$ which are spanned by the eigenfunctions ξ_{kn} , $n = 1, \dots, d_\pi$, corresponding to $s_{k,\pi}$. In other words, the multiplicity of $s_{k,\pi}$ is d_π . From this place, we write π rather than ξ to emphasize our choice of an element ξ from the equivalence class $[\pi]$. Each element $\pi \in \widehat{G}$ can be realised as a finite-dimensional matrix via some choice of a basis in the representation space. Denote by π_{kn} the matrix elements of π , i.e.

$$\pi: G \ni g \mapsto \pi(g) = [\pi_{kn}(g)]_{k,n=1}^{d_\pi} \times \mathbb{C}^{d_\pi \times d_\pi}. \quad (3.41)$$

By the Peter-Weyl theorem (see e.g. [RT10, Theorem 7.5.14]), we have the decomposition

$$L^2(G) = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{n=1}^{d_\pi} \text{span}\{\pi_{kn}\}_{k=1}^{d_\pi}. \quad (3.42)$$

In other words, we can write

$$L^2(G) \ni f = \sum_{\pi \in \widehat{G}} d_\pi \sum_{n=1}^{d_\pi} \sum_{k=1}^{d_\pi} (f, \pi_{kn})_{L^2(G)} \pi_{kn} \in \bigoplus_{\pi \in \widehat{G}} \bigoplus_{n=1}^{d_\pi} \text{span}\{\pi_{kn}\}_{k=1}^{d_\pi}. \quad (3.43)$$

The action of A can be written in the form

$$Af = \sum_{\pi \in \widehat{G}} d_\pi \sum_{n=1}^{d_\pi} \sum_{k=1}^{d_\pi} \sum_{s=1}^{d_\pi} \sigma_A(\pi)_{ns} (f, \pi_{ks})_{L^2(G)} \pi_{kn}, \quad (3.44)$$

This implies where $\sigma_A(\pi)$ is the global matrix symbol of A (cf. [RT13, RT10]).

$$A = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{n=1}^{d_\pi} \sigma_A(\pi). \quad (3.45)$$

Then for the modulus $|A| = \sqrt{AA^*}$ we get

$$|A| = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{n=1}^{d_\pi} |\sigma_A(\pi)|. \quad (3.46)$$

Choosing a representative $\xi \in [\pi]$ from the equivalence class $[\pi]$, we can diagonalise the matrix $|\sigma_A(\pi)|$ as

$$\sigma_{|A|}(\xi) = \begin{pmatrix} s_{1,\xi} & 0 \dots & 0 \\ 0 & s_{2,\xi} \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & s_{d_\xi,\xi} \end{pmatrix}. \quad (3.47)$$

Thus, we obtain

$$|A|f = \sum_{\xi \in \widehat{G}} d_\xi \sum_{n=1}^{d_\xi} s_{k,\xi} \cdot \sum_{k=1}^{d_\xi} (f, \xi_{kn})_{L^2(G)} \xi_{kn}. \quad (3.48)$$

Each $s_{n,\xi}$ is a joint eigenvalue of $|A|$ with the eigenfunctions ξ_{kn}

$$|A|\xi_{k,n} = s_{k,\xi} \xi_{k,n}, \quad n = 1, \dots, d_\xi. \quad (3.49)$$

Since each singular value $s_{k,\xi}$, $k = 1, \dots, d_\xi$, has the multiplicity d_ξ , we obtain

$$E_{[t,+\infty)}(|A|) = \bigoplus_{\xi \in \widehat{G}} \bigoplus_{\substack{k=1, \dots, d_\xi \\ s_{k,\xi} \geq t}} E^{n,\xi}, \quad (3.50)$$

where $E^{n,\xi}$ is the projection to the left-invariant subspace $\text{span}\{\xi_{kn}\}_{k=1}^{d_\xi}$. Consequently, we have

$$\text{Tr}(E_{[t,+\infty)}(|A|)) = \sum_{\xi \in \widehat{G}} d_\xi \sum_{\substack{k=1, \dots, d_\xi \\ s_{k,\xi} \geq t}} 1. \quad (3.51)$$

The proof is now complete. □

REFERENCES

- [ANR15] R. Akyłzhanov, E. Nursultanov, and M. Ruzhansky. Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and L^p - L^q multipliers on compact homogeneous manifolds. *arXiv:1504.07043*, 2015.
- [BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [Bru61] F. Bruhat. Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p -adiques. *Bull. Soc. Math. France*, 89:43–75, 1961.
- [BS88] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.
- [CGM93] M. Cowling, S. Giulini, and S. Meda. L^p - L^q estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces. I. *Duke Math. J.*, 72(1):109–150, 1993.
- [Cow74] M. G. Cowling. *Spaces A_p^q and L^p - L^q Fourier multipliers*. PhD thesis, The Flinders University of South Australia, 1974.
- [CW71] R. R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971. Étude de certaines intégrales singulières.
- [DGR00] A. H. Dooley, S. K. Gupta, and F. Ricci. Asymmetry of convolution norms on Lie groups. *J. Funct. Anal.*, 174(2):399–416, 2000.
- [Dix81] J. Dixmier. *Von Neumann algebras*. Amsterdam ; New York : North-Holland Pub. Co, 1981.
- [DM76] M. Duflo and C. C. Moore. On the regular representation of a nonunimodular locally compact group. *J. Funct. Anal.*, 21(2):209–243, 1976.
- [Ern61] J. Ernest. A decomposition theory for unitary representations of locally compact groups. *Bull. Amer. Math. Soc.*, 67:385–388, 1961.
- [Ern62] J. A. Ernest. A decomposition theory for unitary representations of locally compact groups. *Trans. Amer. Math. Soc.*, 104:252–277, 1962.
- [Fol95] G. B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.
- [FR14] V. Fischer and M. Ruzhansky. Fourier multipliers on graded Lie groups. *arXiv:1411.6950*, 2014.
- [Gra08] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.
- [HL27] G. H. Hardy and J. E. Littlewood. Some new properties of Fourier constants. *Math. Ann.*, 97(1):159–209, 1927.
- [Hör60] L. Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960.
- [Kos81] H. Kosaki. Non-commutative Lorentz spaces associated with a semifinite Von Neumann algebra and applications. *Proc. Japan Acad. Ser. A Math. Sci.*, 57(6):303–306, 1981.
- [Kun58] R. A. Kunze. L_p Fourier transforms on locally compact unimodular groups. *Trans. Amer. Math. Soc.*, 89:519–540, 1958.
- [MR15] M. Mantoiu and M. Ruzhansky. Pseudo-differential operators, Wigner transform and Weyl systems on type I locally compact groups. <http://arxiv.org/pdf/1506.05854v1.pdf>, 2015.
- [MvN36] F. J. Murray and J. von Neumann. On rings of operators. *Ann. of Math. (2)*, 37(1):116–229, 1936.
- [MvN37] F. J. Murray and J. von Neumann. On rings of operators. II. *Trans. Amer. Math. Soc.*, 41(2):208–248, 1937.
- [Naj72] M. Najmark. *Normed algebras*. Wolters-Noordhoff Publishing Groningen, 1972.
- [RT10] M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries. Background analysis and advanced topics*, volume 2 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.

- [RT13] M. Ruzhansky and V. Turunen. Global quantization of pseudo-differential operators on compact Lie groups, $SU(2)$, 3-sphere, and homogeneous spaces. *Int. Math. Res. Not. IMRN*, (11):2439–2496, 2013.
- [RW15] M. Ruzhansky and J. Wirth. L^p Fourier multipliers on compact Lie groups. *Math. Z.*, 280(3-4):621–642, 2015.
- [Seg49] I. E. Segal. Two-sided ideals in operator algebras. *Ann. of Math. (2)*, 50:856–865, 1949.
- [Seg50] I. E. Segal. An extension of Plancherel’s formula to separable unimodular groups. *Ann. of Math.*, 52:272–292, 1950.
- [Seg53] I. Segal. A non-commutative extension of abstract integration. *Ann. of Math.*, 57(3):401–457, 1953.
- [Ter81] M. Terp. L^p spaces associated with von Neumann algebras. *Copenhagen University*, 1981.
- [TK86] F. Thierry and H. Kosaki. Generalized s -numbers of τ -measurable operators. *Pacific J. Math.*, 123(2):269–300, 1986.
- [vN32] J. von Neumann. Über adjungierte Funktionaloperatoren. *Ann. of Math. (2)*, 33(2):294–310, 1932.

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